Seat No. : _______________

OB-143

October-2018

M.Sc., Sem.-II

408 : Mathematics (Old)

(Algebra I)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) Describe the dihedral group D_4 of all the symmetries of a square.

(2) Give an example of an infinite group in which the order of each element is finite. Explain.

OR

(1) State and prove Lagrange's theorem for finite groups.

(2) Prove that a group cannot be expressed as a union of its two proper subgroups.

(B) Attempt any **Four** : **4**

(1) Give an example of an infinite non-Abelian group.

(2) Give an example of an Abelian group which is not cyclic.

(3) True or False : The Dihedral group D_n has a subgroup of index 2.

(4) True or False : The group U(8) is cyclic.

(5) How many generators does the group $\mathbb{Z}_{20}$ have ?

(6) Prove or disprove : The group $(\mathbb{R}, +)$ is cyclic.

2. (A) Answer the following questions : **14**

(1) Prove that every permutation of a finite set can be expressed a cycle or as a product of disjoint cycles.

(2) Prove that $Z(S_n) = \{e\}$, if $n \geq 3$.

OR

(1) State and prove Cayley's theorem.

(2) Prove that the group $Aut\ (\mathbb{Z}_{100})$ is isomorphic to U(100).

(B) Attempt any **Four** : **4**

(1) How many elements of order 5 are there in S_6?

(2) Let $\beta = (123)\,(145)$. Write β^{99} in cycle form.

(3) What is the order of the group $Aut\,(\mathbb{Z}_{12})$? Explain.

(4) Prove that any infinite cyclic group is isomorphic to $\mathbb{Z}$.

(5) How many elements of order 2 are there in the group $\mathbb{Z}_{100} \oplus \mathbb{Z}_{400}$? Explain.

(6) True or False : The group $\mathbb{Z}_8 \oplus \mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_{96}$.

3. (A) Answer the following questions : **14**

(1) State and prove the first isomorphism theorem.

(2) State (only) the fundamental theorem of finite Abelian groups. Determine the isomorphic classes of Abelian group of order 100.

OR

(1) Define homomorphism. If ϕ is a homomorphism from group G to K, prove that $|\phi(g)|$ divides $|g|$.

(2) Prove that any group of order 35 is cyclic.

(B) Attempt any **Three** : **3**

(1) True or False: G/H is Abelian iff G is Abelian. Justify.

(2) If H is a subgroup of G of index 2, prove that H is normal in G.

(3) Give an example of a subgroup of S_3 that is not normal in S_3.

(4) Define : Normalizer N(H) and Centralizer C(H) of a subgroup H.

(5) Prove that the kernel of a homomorphism is a normal subgroup.

4. (A) Answer the following questions : **14**

(1) State and prove Sylow's first theorem.

(2) Define simple groups. Prove that A_5 is simple.

OR

(1) State and prove Sylow's third theorem.

(2) State and prove non-simplicity test.

(B) Attempt any **Three** : **3**

(1) True or False : Any group of prime order is simple.

(2) True or False : Any group of order p^3 is Abelian. Justify.

(3) Prove that there is no simple group of order 210.

(4) True or False : If $|G| = p^4$, then its centre $Z(G)$ is non-trivial.

(5) Every non-Abelian group is simple. True or False ?

Seat No. : ______________

OB-143

October-2018

M.Sc., Sem.-II

408 : Mathematics (New) (Repeater)

(Real Analysis)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) (i) State without proof Egorov's theorem. Verify Egorov's theorem for the sequence $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = 3x^{2n}$. **7**

(ii) If the sequence of measurable functions $f_n(x)$ converges to $f(x)$ almost everywhere on a bounded measurable set E, prove that $f_n \Rightarrow f$, i.e. the sequence $f_1(x), f_2(x), f_3(x)\ldots$ converges in measure to $f(x)$. **7**

OR

(i) State without proof Luzin's theorem. **7**

Let $g : [0, 1] \to \mathbb{R}$ be the function given by

$$g(x) = 3, \text{ if } 0 \le x \le \frac{1}{4},$$

$$g(x) = 0, \text{ if } \frac{1}{4} < x \le 1.$$

Verify Luzin's theorem for the function $g(x)$.

(ii) State without proof Weierstrass' theorem. Express $\cos^2 x$ as a trigonometric polynomial. **7**

(B) Answer any **four** : **4**

(i) True or false : If f_n converges in measure to f, then every subsequence of $\{f_n\}$ converges in measure to f. (Do not prove).

(ii) Define : convergence in measure.

(iii) True or false : f_n converges in measure to f, and also f_n converges in measure to g, then f = g almost everywhere. (Do not prove).

(iv) If $f(x) = x^3$ is defined on [0, 1], find $B_2(x)$ for this function $f(x)$.

(v) Give an example of a trigonometric polynomial of the 2nd order.

(vi) Give (without proof) a convergent series of positive terms.

2. (A) (i) Define (without proof) the norm of L_2, the set of all square summable functions defined on a closed interval [a, b]. Show that if a sequence $\{f_n(x)\}$ in L_2 converges in the mean to the function $f(x)$, then it converges in measure to $f(x)$. **7**

(ii) Show that $x = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right)$ belongs to L_2. Find the norm $\| x \|$. **7**

OR

(i) Show that the set of bounded measurable functions is everywhere dense in L_2 [a, b]. **7**

(ii) Define the space L_p, the set of P^{th} power summable functions defined on a closed interval [a, b]. ($p \geq 1$). **7**

Show that the sum of two functions in L_p is again a function in L_p.

(B) Answer any **four** : **4**

(i) State (without proof) Holder's inequality.

(ii) State (without proof) Minkowski's inequality.

(iii) True or false : Every square summable function is summable. (Do not prove).

(iv) True or false : The class of polynomials is everywhere dense in L_2 [a, b]. (Do not prove).

(v) Let $x = (1, 0, 0, 0, 0, \ldots)$. Is $x \in l_2$?

(vi) If p = 5, find q, the index conjugate to p.

3. (A) (i) Show that the set of points of discontinuity of an increasing function f(x) defined on [a, b] is at most countable. **7**

(ii) Define a function of finite variation on [a, b]. State (without proof) a theorem completely describing a function of finite variation in terms of increasing functions. **7**

OR

(i) Define a derived number of a function f(x) at a point x_o. **7**

Show that if a function f(x) defined on [a, b] has a derivative $f'(x_0)$ at the point x_0 in [a, b], then all derived numbers of f(x) at this point are equal.

(ii) Define an absolutely continuous function f(x) on [a, b]. Show that the sum of two absolutely continuous functions is also absolutely continuous. **7**

(B) Answer any **three** : **3**

(i) True or false : Every absolutely continuous function defined on [0, 1] has finite variation.

(ii) True or false : The cantor function is absolutely continuous on [0, 1]. (Do not prove).

(iii) Write $\sin x$ as the difference of two continuous increasing functions defined on $[0, \pi]$.

(iv) Define $f(x) = x$, if x is rational,

$= 3x$, if x is irrational.

Find a derived number of $f(x)$ at $x_0 = 0$.

(v) True or false : $f(x) = x^2$ defined on [0, 1] is a function of finite variation.

4. (A) (i) Determine the Fourier series of the 2π-periodic function

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ 1, & 0 < t < \pi \end{cases}$$

Deduce from it the value of the infinite sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$ (Justify). **7**

(ii) State Parseval's equality for a function $f(x) \in L_2[-\pi, \pi]$. (Do not prove). **7**

OR

(i) State and prove the Riemann-Lebesgue Theorem. **7**

(ii) State (without proof) a necessary and sufficient condition for a function $\phi(x)$ to be the indefinite integral of a summable function. **7**

(B) Answer any **three** : **3**

(i) Define an odd function f on $[-\pi, \pi]$.

(ii) Define an even function f on $[-\pi, \pi]$.

(iii) Give an example of an odd function defined on $[-\pi, \pi]$.

(iv) True or false : $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is the Fourier series of some function in $L_2[-\pi, \pi]$.

(v) Show that $\sin x$ and $\cos x$ are orthogonal in $L_2[-\pi, \pi]$.